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The Lyapunov Spectrum of a Continuous Product of Random Matrices

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We present a functional integration method for the averaging of continuous products $\hat{P}_t$ of $N \times N$ random matrices. As an application, we compute exactly the statistics of the Lyapunov spectrum of $\hat{P}_t$. This problem is relevant to the study of the statistical properties of various disordered physical systems, and specifically to the computation of the multipoint correlators of a passive scalar advected by a random velocity field. Apart from these applications, our method provides a general setting for computing statistical properties of linear evolutionary systems subjected to a white-noise force field.

KEY WORDS: Lyapunov exponents; random matrices; functional integral; disordered systems; passive scalar; Gauss decomposition; loop groups.

1. INTRODUCTION

In this work we give a detailed exposition of a functional integral method for the averaging of time-ordered exponentials of $N \times N$ random matrices which has found several applications in the study of the statistical properties of disordered systems.

The method was introduced by Kolokolov[(2)] in the $N = 2$ case in order to compute the partition function of the Heisenberg ferromagnet, and was thereafter applied to the study of one-dimensional Anderson localization[(3)] and to some problems of mesoscopic physics.[(4)] Later, the same technique[(6, 7)] was used to obtain analytical results in the problem of a passive scalar advected by a two-dimensional random velocity field. The approach

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of refs. 6 and 7 was then extended to the more general N-dimensional case in ref. 1.

In the present work we present the method in its full generality and show that it allows one to compute exactly the statistics of the whole Lyapunov spectrum of the matrix $\hat{P}_t$ describing the time evolution of a linear system subjected to a white-noise force field. Such a spectrum is relevant[8] in the computation of the multipoint correlators of the passive scalar (as well as in the computation of the correlators of passive vectors and tensors); in this case the matrix $\hat{P}_t$ describes the time evolution of particles in a turbulent fluid linearized around a given trajectory.

However, our setting presents a high degree of generality and therefore a wider range of applications. In particular, our formalism can be naturally applied to the study of N-level quantum mechanical systems affected by a random noise, and it is complementary to the supersymmetric approach[14, 15] to the problem of $N/2$ channel localization in a disordered conductor.

2. AVERAGES OF TIME-ORDERED EXPONENTIALS

Let us start with the problem of computing Gaussian averages of the form

$$\langle \mathscr{F}[\hat{P}_t] \rangle = \frac{1}{\mathscr{N}} \int \mathscr{D}\hat{X} \exp(-S[\hat{X}])\, \mathscr{F}[\hat{P}_t] \tag{1}$$

where $\hat{X}(s)$, for $0 \leqslant s \leqslant T$, is a traceless $N \times N$ hermitian matrix,

$$\mathscr{D}\hat{X} \equiv \prod_{0 \leqslant s \leqslant T} \prod_{i<j} dX_{ij}(s)\, dX_{ji}(s) \prod_i dX_{ii}(s) \tag{2}$$

is the Feynman–Kac measure, $\mathscr{N}$ is chosen in such a way that $\langle 1 \rangle = 1$,

$$S[\hat{X}] = \frac{1}{4D} \int_0^T \operatorname{Tr} \hat{X}^2(s)\, ds \tag{3}$$

and $\hat{P}_t$ is the time-ordered exponential

$$\hat{P}_t = \mathscr{T} \exp\left(\int_0^t \hat{X}(s)\, ds \right) \tag{4}$$

such that $\mathbf{r}(t) = \hat{P}_t \mathbf{r}_0$ is the general solution of the linear problem $\dot{\mathbf{r}} = \hat{X}\mathbf{r}$, $\mathbf{r}(0) = \mathbf{r}_0$, and

$$\dot{\hat{P}}_t \hat{P}_t^{-1} = \hat{X} \tag{5}$$

We shall now introduce a set of "collective" integration variables in order to reexpress (4) in a more tractable form. At the same time, we shall chose the new variables in such a way that (3) is quadratic and that the Jacobian determinant of the functional transformation is particularly simple.

As a first step, let us Gauss-decompose the matrix $\hat{P}_t$:

$$\hat{P}_t = (\mathbf{1} + \hat{\phi}(t)) \cdot \exp(\hat{\tau}(t)) \cdot (\mathbf{1} + \hat{\theta}(t)) \tag{6}$$

where

$$\begin{aligned} &\phi_{ij}(t) \equiv 0, \qquad i \leqslant j \\ &\theta_{ij}(t) \equiv 0, \qquad i \geqslant j \\ &\tau_{ij}(t) \equiv \tau_i(t)\, \delta_{ij} \end{aligned} \tag{7}$$

$$\tau_N(t) \equiv -\sum_{j=1}^{N-1} \tau_j(t) \tag{8}$$

Moreover, in order to ensure the equality $\hat{P}_0 = \mathbf{1}$ we shall impose

$$\hat{\phi}(0) = \mathbf{0}, \qquad \hat{\theta}(0) = \mathbf{0} \tag{9}$$

We now reexpress the "local" degrees of freedom $X_{ij}(t)$ in terms of the global ones $\phi_{ij}(t)$, $\tau_{ij}(t)$, $\theta_{ij}(t)$. This can be accomplished by making use of the basis $\hat{e}_{ij}$ of the matrix algebra, which is defined by $(\hat{e}_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and satisfies the commutation rules

$$[\hat{e}_{ij}, \hat{e}_{kl}] = \delta_{jk}\hat{e}_{il} - \delta_{il}\hat{e}_{kj} \tag{10}$$

In particular, one has

$$\begin{aligned} \hat{e}_{ii}\hat{e}_{kl} &= \hat{e}_{kl}(\hat{e}_{ii} + \delta_{ik} - \delta_{il}) \\ e^{\hat{\tau}}\hat{e}_{ij}e^{-\hat{\tau}} &= e^{\tau_i - \tau_j}\hat{e}_{ij} \end{aligned} \tag{11}$$

From these relations the desired expression for X_{ij} readily follows:

$$\begin{aligned} X_{ij} = {} & \dot{\phi}_{ij} + \sum_k \dot{\phi}_{ik}\tilde{\phi}_{kj} \\ & + \dot{\tau}_i\delta_{ij} + \phi_{ij}\dot{\tau}_j + \dot{\tau}_i\tilde{\phi}_{ij} + \sum_k \phi_{ik}\dot{\tau}_k\tilde{\phi}_{kj} \\ & + A_{ij} + \sum_k (\phi_{ik}A_{kj} + A_{ik}\tilde{\phi}_{kj}) + \sum_{k,l} \phi_{ik}A_{kl}\tilde{\phi}_{lj} \end{aligned} \tag{12}$$

where

$$
\begin{aligned}
A_{ij} &\equiv e^{\tau_i - \tau_j} \sum_k \dot{\theta}_{ik}(\delta_{kj} + \tilde{\theta}_{kj}) \\
\tilde{\phi}_{ij} &\equiv -\phi_{ij} + \sum_k \phi_{ik}\phi_{kj} - \sum_{k,l} \phi_{ik}\phi_{kl}\phi_{lj} + \cdots
\end{aligned}
\tag{13}
$$

and a similar definition holds for $\tilde{\theta}_{ij}$ [for any fixed N the sum in (13) is finite, since $\hat{\phi}$ is a nilpotent matrix; the same is true for $\hat{\theta}$].

Substituting (12) in (3), one obtains

$$
\tfrac{1}{2}\,\mathrm{Tr}\,\dot{X}^2 = \tfrac{1}{2}\sum_{j=1}^{N} \dot{\tau}_j^2 + \sum_{i,j}\left(\dot{\phi}_{ij} + \sum_{k=1}^{N} \tilde{\phi}_{ik}\dot{\phi}_{kj}\right) e^{\tau_j - \tau_i}\left(\sum_{k=1}^{N} \dot{\theta}_{jl}\tilde{\theta}_{li} + \dot{\theta}_{ji}\right) \tag{14}
$$

The form of (14) suggests the introduction of the new variables

$$
\begin{aligned}
\bar{\phi}_{ij} &= \sum_{k,l} \dot{\theta}_{ik}(\delta_{kl} + \tilde{\theta}_{kl})(\delta_{lj} + \dot{\phi}_{lj} e^{\tau_j - \tau_l})\, e^{\tau_i - \tau_j}, \qquad i < j \\
\bar{\phi}_{ij} &\equiv 0, \qquad i \geqslant j
\end{aligned}
\tag{15}
$$

so that

$$
\tfrac{1}{2}\,\mathrm{Tr}\,\dot{X}^2 = \tfrac{1}{2}\sum_{j=1}^{N} \dot{\tau}_j^2 + \sum_{i,j} \dot{\phi}_{ij}\bar{\phi}_{ji} \tag{16}
$$

Relation (15) can be inverted giving

$$
\dot{\theta}_{ij} = \sum_{k,l} \bar{\phi}_{ik} e^{\tau_k - \tau_i}(\delta_{kl} + \phi_{kl} e^{\tau_l - \tau_k})(\delta_{lj} + \theta_{lj})\,\chi_{li} \tag{17}
$$

where

$$
\chi_{ij} \equiv 1 - \bar{\chi}_{ij} \equiv \begin{cases} 1, & i > j \\ 0, & i \leqslant j \end{cases} \tag{18}
$$

Through (17) one can reexpress the θ_{ij} as functions of the new variables ϕ_{ij}, τ_i and $\bar{\phi}_{ij}$ in a recursive way, thanks to the "triangular" form of the equation. For instance, for $N = 3$ one gets

$$
\begin{aligned}
\theta_{23}(t) &= \int_0^t \bar{\phi}_{23}(s)\, e^{-\tau_1(s) - 2\tau_2(s)}\, ds \\
\theta_{12}(t) &= \int_0^t A(s)\, ds, \qquad \text{where} \quad A = (\bar{\phi}_{12} + \bar{\phi}_{13}\phi_{32})\, e^{\tau_2 - \tau_1} \\
\theta_{13}(t) &= \int_0^t [A(s)\,\theta_{23}(s) + \bar{\phi}_{13}(s)\, e^{-2\tau_1(s) - \tau_2(s)}]\, ds
\end{aligned}
\tag{19}
$$

As a matter of fact, for any fixed i the $N-i$ functions θ_{ij} can be expressed through the $N-i-1$ functions $\theta_{i+1,j}$ and the remaining variables by means of a single quadrature. This is an important point, since for practical calculations $\hat{P}_t$ has to be reexpressed in terms of the new variables $\hat{\phi}$, $\dot{\tau}$, $\hat{\bar{\phi}}$.

We must now substitute $\hat{X}(s)$ as an integration variable in the functional integral (1) with the new variables $\hat{\phi}(s)$, $\dot{\tau}(s)$, $\hat{\bar{\phi}}(s)$. Again using the commutation rules (10), and renaming $\dot{\tau}_i \equiv \rho_i$ for convenience, we finally get

$$\begin{aligned} X_{ij} &= \bar{\phi}_{ij} + \sum_k \phi_{ik}\bar{\phi}_{kj}, \qquad i<j \\ X_{ii} &= \rho_i + \sum_k (\phi_{ik}\bar{\phi}_{ki} - \bar{\phi}_{ik}\phi_{ki}), \qquad i=1,\ldots,N \end{aligned} \tag{20}$$

$$\begin{aligned} X_{ij} &= \phi_{ij}\rho_j + \rho_i\tilde{\phi}_{ij} + \dot{\phi}_{ij} \\ &\quad + \sum_k (\phi_{ik}\rho_k\tilde{\phi}_{kj} + \dot{\phi}_{ik}\tilde{\phi}_{kj} + \phi_{ik}\bar{\phi}_{kj} - \bar{\phi}_{ik}\phi_{kj}) \\ &\quad - \sum_{k,l} (\bar{\chi}_{jk}\phi_{ik}\bar{\phi}_{kl}\phi_{lj} + \bar{\chi}_{li}\bar{\phi}_{ik}\phi_{kl}\tilde{\phi}_{lj}) \\ &\quad - \sum_{k,l,m} \bar{\chi}_{mk}\phi_{ik}\bar{\phi}_{kl}\phi_{lm}\tilde{\phi}_{mj}, \qquad i>j \end{aligned} \tag{21}$$

In ref. 1 the $N=3$ case was explicitly considered. We observe that the matrix elements of $\hat{X}(t) = \dot{\hat{P}}_t\hat{P}_t^{-1}$ transform as

$$X_{ij}(t) \to e^{\sigma_{ij}} X_{ij}(t) \tag{22}$$

under the global gauge transformation

$$\phi_{ij}(t) \to e^{\sigma_{ij}}\phi_{ij}(t), \qquad \bar{\phi}_{ij}(t) \to e^{\sigma_{ij}}\bar{\phi}_{ij}(t), \qquad \rho_i(t) \to \rho_i(t) \tag{23}$$

where σ_{ij} satisfies $\sigma_{ik} + \sigma_{kj} = \sigma_{ij}$, $\sigma_{ij} = -\sigma_{ji}$.

As the last step, we must compute the Jacobian determinant of the functional transformation (21). Notice first that through the shift given by

$$\begin{aligned} \bar{\phi}_{ij} &\to \bar{\phi}_{ij} - \sum_{k=1}^{N} \phi_{ik}\bar{\phi}_{kj} \\ \rho_i &\to \rho_i - \sum_{k=1}^{N} (\phi_{ik}\bar{\phi}_{ki} - \bar{\phi}_{ik}\phi_{ki}) \end{aligned} \tag{24}$$

(which has Jacobian $\mathcal{J}' = 1$) one can reduce to the computation of

$$\mathrm{Det}\left(\frac{\delta \hat{X}_-}{\delta \hat{\phi}\, \delta \hat{\rho}\, \delta \hat{\bar{\phi}}}\right) \tag{25}$$

where $\hat{X}_-$ is the strictly lower triangular part of $\hat{X}$. The Jacobian (25) can be computed by means of the standard regularization procedure[2]

$$\begin{gathered} \hat{\phi}_n = \hat{\phi}(t_n), \qquad \hat{\rho}_n = \hat{\rho}(t_n), \qquad \hat{\bar{\phi}}_n = \hat{\bar{\phi}}(t_n) \\ t_n = hn, \qquad n = 1, \ldots, M, \quad h = T/M \to 0, \quad M \to +\infty \\ \dot{\hat{\phi}}(t) \to \frac{\hat{\phi}_n - \hat{\phi}_{n-1}}{h}, \qquad \hat{\phi}(t) \to \frac{\hat{\phi}_n + \hat{\phi}_{n-1}}{2} \end{gathered} \tag{26}$$

giving

$$\mathcal{J} \propto \exp\left(\sum_{j=1}^{N-1} (N-j) \int_0^T \rho_j(s)\, ds\right) \tag{27}$$

Applying now the variable transformation $\hat{X} \to (\hat{\phi}, \hat{\rho}, \hat{\bar{\phi}})$, one sees that the functional integral (1) reduces to

$$\langle \mathcal{F}[\hat{P}_t] \rangle = \frac{1}{\mathcal{N}'} \int \mathcal{D}\hat{\phi}\, \mathcal{D}\hat{\rho}\, \mathcal{D}\hat{\bar{\phi}} \exp(-S'[\hat{\phi}, \hat{\rho}, \hat{\bar{\phi}}])\, \mathcal{F}[(\mathbf{1} + \hat{\phi})\, e^{\hat{\tau}}(\mathbf{1} + \hat{\theta})] \tag{28}$$

where $\hat{\theta} = \hat{\theta}[\hat{\phi}, \hat{\rho}, \hat{\bar{\phi}}]$ is obtained by solving (17), $\hat{\tau} = \int_0^T \hat{\rho}(s)\, ds$, the ρ_i are constrained by (8), $\mathcal{N}'$ is the normalization factor, and

$$S' = \frac{1}{2D} \int_0^T \left(\frac{1}{2} \sum_{k=1}^{N} \rho_k^2 + \sum_{i,j} \dot{\phi}_{ij} \bar{\phi}_{ji} - 2D \sum_{k=1}^{N-1} (N-k)\, \rho_k\right) ds \tag{29}$$

In (1) the functional integration is constrained to the surface

$$\Gamma_0 = \{X_{ij}(s) = X_{ji}^*(s),\, 0 \leqslant s \leqslant T\} \tag{30}$$

In refs. 2 and 7 it was shown, using the Cauchy theorem, that whenever $\mathcal{F}$ is holomorphic in the matrix elements X_{ij} one can modify the integration surface Γ_0 to the homotopically equivalent

$$\Gamma_1\colon \{\phi_{ij}(s) = \bar{\phi}_{ji}^*(s),\ \mathrm{Im}\, \rho_i(s) = 0,\, 0 \leqslant s \leqslant T\} \tag{31}$$

without affecting the value of the integral. This means that in (28) $\hat{\bar{\phi}}$ may be regarded as the Hermitian conjugate of $\hat{\phi}$.

We remark that expressions similar to (21) were obtained in the framework of conformal field theory.[16] However, the explicit forms of the variables $\bar{\phi}$ and of the Jacobian $\mathscr{J}$, which are essential for any physical application of our method, were not computed there.

3. THE LYAPUNOV SPECTRUM

We shall now define the Lyapunov exponents λ_j, $j = 1,..., N$, according to the relation[9]

$$\lambda_1 + \cdots + \lambda_k = \frac{1}{T} \log \mathrm{Vol}(\hat{P}_T \mathbf{v}_1, ..., \hat{P}_T \mathbf{v}_k) \tag{32}$$

where $\mathbf{v}_1, ..., \mathbf{v}_k$ is an orthonormal set of vectors generating a unitary k-volume. For the sake of definiteness we shall choose $\mathbf{v}_j = \mathbf{e}_j$, where $(\mathbf{e}_j)_i = \delta_{ij}$. One has

$$\mathrm{Vol}(\hat{P}_T \mathbf{e}_1, ..., \hat{P}_T \mathbf{e}_k) = \left[\sum_{a=1}^{l_k} \Delta_a^2(\hat{M}_k) \right]^{1/2} \tag{33}$$

where $\Delta_a(\hat{M}_k)$, $a = 1,..., l_k$, $l_k \equiv \binom{n}{k}$ are the $k \times k$ minors of the $n \times k$ matrix $\hat{M}_k = [\hat{P}_T \mathbf{e}_1, ..., \hat{P}_T \mathbf{e}_k]$. Let $\boldsymbol{\phi}_j$, $\mathbf{p}_j$, $j = 1,..., N$, be the vectors defined by $(\boldsymbol{\phi}_j)_i = \delta_{ij} + \phi_{ij}$, $(\mathbf{p}_j)_i = (\hat{P}_T)_{ij}$. Then

$$\begin{aligned} \mathbf{p}_j &= \sum_{i,k} (\delta_{ik} + \phi_{ik})\, e^{\tau_k} (\delta_{kj} + \theta_{kj})\, \mathbf{e}_i \\ &= \sum_{k=1}^{N} e^{\tau_k} (\delta_{kj} + \theta_{kj})\, \boldsymbol{\phi}_k \\ &= e^{\tau_j} \boldsymbol{\phi}_j + \sum_{k<j} \theta_{kj} e^{\tau_k} \boldsymbol{\phi}_k \end{aligned} \tag{34}$$

From (34) and the multilinearity of determinants it follows that

$$\begin{aligned} \Delta_a(M_k) &= \Delta_a[\mathbf{p}_1, ..., \mathbf{p}_n] \\ &= \Delta_a[e^{\tau_1} \boldsymbol{\phi}_1, ..., e^{\tau_k} \boldsymbol{\phi}_k] \\ &= e^{\tau_1 + \cdots + \tau_l} \left(\delta_{a,1} + \sum_{r_{ij} \geq 1} c_{ij}^{a,k} \phi_{ij}^{r_{ij}} \right) \end{aligned} \tag{35}$$

where Δ_1 is the minor obtained from the first k rows of M_k, r_{ij} are strictly positive integers, and $c_{ij}^{a,k}$ are integer coefficients with $c_{ij}^{1,k} \equiv 0$. One has then

$$\lambda_1 + \cdots + \lambda_k = \frac{1}{T}[\tau_1(T) + \cdots + \tau_k(T))] + \frac{1}{2T}\log\left[1 + \sum_{a=2}^{l_k}\left(\sum_{r_{ij}\geq 1} c_{ij}^{a,k}\phi_{ij}^{r_{ij}}(T)\right)^2\right] \tag{36}$$

and

$$\lambda_k = \frac{1}{T}\int_0^T \rho_k\, dt + \frac{1}{2T}\{\log[1 + f_k(\hat{\phi})] - \log[1 + f_{k-1}(\hat{\phi})]\} \tag{37}$$

where $1 + f_k(\hat{\phi})$ is the argument of the logarithm in (36).

Let us now compute the probability distribution function for λ_k. The form of (29) implies that the ϕ-dependent terms in (37) give no contribution, since they do not contain the conjugate variables $\bar{\phi}_{ij}$.

We are therefore left with $N-1$ Gaussian integrations over $\rho_1,..., \rho_{k-1}$, $\rho_{k+1},..., \rho_N$ which give the following exact result for the statistics of ρ_k:

$$\mathscr{D}\rho_k \exp\left(-\frac{N}{4D(N-1)}\int_0^T [\rho_k(s) - \bar{\lambda}_k]^2\, ds\right) \tag{38}$$

where

$$\bar{\lambda}_k = D(N-2k+1), \qquad k = 1,..., N \tag{39}$$

The probability distribution function $p(\lambda_k;T)$ of the kth Lyapunov exponent λ_k is then

$$p(\lambda_k; T) = \frac{1}{2}\left(\frac{NT}{\pi D(N-1)}\right)^{1/2} \exp\left(-\frac{NT}{4D(N-1)}(\lambda_k - \bar{\lambda}_k)^2\right) \tag{40}$$

The Lyapunov exponents λ_k are statistically dependent due to the constraint (8) and their joint distribution function has a generalized Gaussian form.[3] We have thus obtained a complete knowledge of the statistics of the Lyapunov spectrum of the matrix $\hat{P}_t$. This has an essential application to the problem of the computation of the multipoint correlators of a passive scalar advected by a random velocity field (see the Appendix).

[3] The Gaussian distribution of the Lyapunov exponent in the $N=2$ case was obtained in the context of the passive scalar problem in ref. 6.

4. CONCLUSIONS

In this work we gave a detailed exposition in the general $N \times N$ case of a functional integral method for the averaging of time-ordered exponentials of random matrices which has found several applications in the study of the statistical properties of disordered physical systems.[1–4, 6–8] and we have shown how the statistics of the Lyapunov spectrum of a linear evolutionary process can be computed exactly. As a matter of fact, our method provides a general setting for computing the statistics of linear evolutionary systems subjected to a white-noise force field.

We conclude with some remarks. The definition of the Lyapunov exponents as the logarithmic rate of growth of a k-dimensional parallelepiped [see Eq. (32) and ref. 9] is the most natural from a physical point of view, e.g., in the passive scalar problem. Generally speaking these exponents do not coincide with the logarithms of the eigenvalues of the evolution matrix $\hat{P}_t$. The statistics of the eigenvalues of a similar evolution matrix was studied in refs. 10–13. Our method, however, allows one to obtain more-detailed statistical information about the evolution of initial vectors and to compute nontrivial correlation functions of their components. For an application to the passive scalar problem see ref. 8.

Lastly, we would like to remark that a more refined application of the functional integral method we described allowed the effective solution of the more difficult case of "colored" noise.[7]

APPENDIX

The method given in this work has a direct application to the computation of the statistics of a scalar passively advected by a random velocity field. In order to illustrate this point we will briefly recall here the terms of the problem. For more detail see refs. 5–8.

The evolution of a scalar field $\theta(\mathbf{r}, t)$ passively advected by a velocity field $\mathbf{v}(\mathbf{r}, t)$ and generated by a source $\phi(\mathbf{r}, t)$ is given by

$$\dot{\theta} + \mathbf{v} \cdot \nabla\theta = \phi \tag{A.1}$$

If we impose on (A.1) the asymptotic condition $\theta(\mathbf{r}, -\infty) = 0$, we get the solution

$$\theta(\mathbf{r}, t) = \int_0^{+\infty} \phi(\mathbf{R}(\mathbf{r}, t-s), t-s)\, ds \tag{A.2}$$

saying that $\theta(\mathbf{r}, t)$ is completely determined in terms of the trajectories $\mathbf{R}(\mathbf{r}_0, t)$ of the fluid particles:

$$\dot{\mathbf{R}} = \mathbf{v}(\mathbf{R}, t), \qquad \mathbf{R}(\mathbf{r}_0, 0) = \mathbf{r}_0 \tag{A.3}$$

Let us now take ϕ and $\mathbf{v}$ to be random, δ-correlated-in-time fields. The source ϕ will be assumed to be spatially correlated on a scale L. The velocity field $\mathbf{v}$ might be multiscale, with smallest scale larger than or of the order of L. The statistics of ϕ and $\mathbf{v}$ will be assumed to be spatially isotropic.

Generally speaking, one is interested in computing equal-time correlators of the form $\langle \theta(\mathbf{r}_1, 0)\, \theta(\mathbf{r}_2, 0)\rangle$ for $|\mathbf{r}_2 - \mathbf{r}_1| \ll L$. From the isotropicity of the statistics of ϕ and $\mathbf{v}$ it follows that such quantities are rotation-invariant. Moreover, (A.2) implies that the statistics of θ is completely determined in terms of the statistics of the trajectories (A.3).

In order to subtract the effect of sweeping, let us choose a reference frame locally comoving with one of the fluid particles.[5–7] We can then locally linearize (A.3), obtaining

$$\dot{\mathbf{R}} \simeq \hat{\sigma}(t)\, \mathbf{R} \tag{A.4}$$

where $\sigma_{ij} \equiv \partial v_j / \partial r_i$, the matrix of velocity derivatives, will be taken to be a random Gaussian process. In the general case we have $\hat{\sigma} = \hat{R} + \hat{S}$, where $\hat{R}$ is the antisymmetric part of $\hat{\sigma}$, inducing a rotation of the passive scalar blob, and $\hat{S}$ is the symmetric part, representing the stretching of the unit blob. We will consider the case of an incompressible fluid, so $\mathrm{Tr}\hat{S} = 0$.

More specifically, let us consider the following statistics of $\hat{\sigma}$:

$$\begin{aligned} \mathscr{D}\mathscr{M}[\hat{\sigma}] &= \mathscr{D}\hat{\sigma} \exp\left(-\frac{1}{2D_s} \int_0^T \mathscr{L}\, dt \right) \\ \mathscr{L} &= \frac{1}{2}\left(\mathrm{Tr}\, \hat{S}^2 - \frac{D_s}{D_r} \mathrm{Tr}\, \hat{R}^2 \right) = \frac{1}{2} \mathrm{Tr}\left[\hat{S} + i \left(\frac{D_s}{D_r} \right)^{1/2} \hat{R} \right]^2 \end{aligned} \tag{A.5}$$

Since we are interested in rotation-invariant quantities the final result shall be independent of D_r. This arbitrariness allows us to substitute $(D_s/D_r)^{1/2} \to -i$, $\hat{R} \to i\hat{R}$, $\hat{\sigma} \to \hat{X} = \hat{S} + i\hat{R}$,[1] and thus to consider a generic traceless Hermitian matrix $\hat{X}$ with averaging weight $\exp\{-[1/(2D_s)] \int \frac{1}{2} \mathrm{Tr}\, \hat{X}^2\}$ in the place of the generic traceless real matrix $\hat{\sigma}$ with the averaging weight (A.5): this allows one to refer to the results of Section 2.

From rotational invariance it follows that the statistics of θ is completely determined in terms of the statistics of the Lyapunov spectrum of the matrix $\hat{P}_t$ defined by

$$\dot{\hat{P}}_t = \hat{X}\hat{P}_t, \qquad \hat{P}_0 = \mathbf{1} \tag{A.6}$$

This reduces the problem to that studied in the preceding sections. The Gaussian statistics of the Lyapunov exponents agrees with an old result[17] about the Gaussian statistics of a line element in a δ-correlated-in-time velocity field.

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